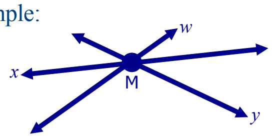
Points of Concurrency

Concurrent lines are three or more lines that intersect at the same point. The mutual point of intersection is called the **point of concurrency** .

Example:



M is the point of concurrency of lines w, y, and x.

The Four Centers of a Triangle

In a triangle, the following sets of lines are concurrent:

- The three **medians** .
- The three **altitudes** .
- The **perpendicular bisectors** of each of the three sides of a triangle.
- The three **angle bisectors** of each angle in the triangle.

Concurrency of the Medians

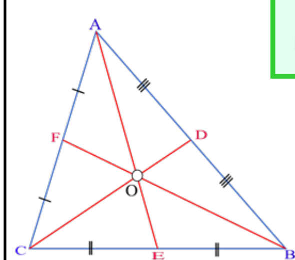
The **median** of a triangle is the line segment that joins the vertex to the midpoint of the opposite side of the triangle.

The three medians of a triangle are concurrent in a point that is called the **centroid**.

There is a special relationship that involves the line segments when all of the three medians meet.

The distance from each vertex to the centroid is two-thirds of the length of the entire median drawn from that vertex

Let's Take a Look at the Diagram....



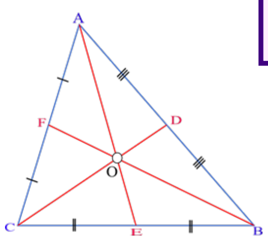
O is the centroid of ΔABC, points D, F, and E are midpoints.

$$\overline{AO} = \frac{2}{3}\overline{AE}$$

$$\overline{BO} = \frac{2}{3}\overline{BF}$$

$$\overline{CO} = \frac{2}{3}\overline{CD}$$

In addition, the distance from each centroid to the opposite side (midpoint) is one-third of the distance of the entire median.



O is the centroid of ΔABC, points D, F, and E are midpoints.

$$\overline{OF} = \frac{1}{3}\overline{BF}$$

$$\overline{OE} = \frac{1}{3}\overline{AE}$$

$$\overline{OD} = \frac{1}{3}\overline{CD}$$

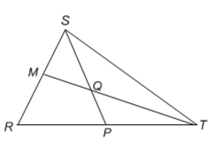
The centroid also divides the median into two segments in the ratio 2:1, such that:

$$\frac{AO}{OE} = \frac{2}{1} \text{ and } \frac{CO}{OD} = \frac{2}{1} \text{ and } \frac{BO}{OF} = \frac{2}{1}$$

If you notice, the bigger part of the ratio is the segment that is drawn from the vertex to the centroid. The smaller part of the median is always the part that is drawn from the centroid to the midpoint of the opposite side.

When working with these ratios, it is important to never mix the two up!!!

Examples:

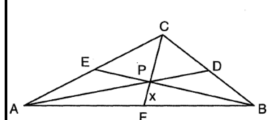


1) In ΔRST, medians $\overline{TM}$ and $\overline{SP}$ are concurrent at point Q. If TQ = 3x-1 and QM = x+1, what is the length of median TM?

2) In ΔABC, points J, K, and L are the midpoints of sides AB, BC, and AC, respectively. If the three medians of the triangle intersect at point P and the length of LP is 6, what is the length of BL?

3) In triangle ABC, medians AD, BE, and CF are concurrent at point P. If AD = 24 inches, find the length of AP.

More Examples!!!



4) In the diagram Jose found centroid P by constructing the three medians. He measured CF and found it to be 6 inches. If PF = x, then what equation can be used to find the value of x?

(1) x + x = 6 (2) 2x + x = 6 (3) 3x + 2x = 6 (4) x + (2/3)x = 6

The Coordinates of the Centroid

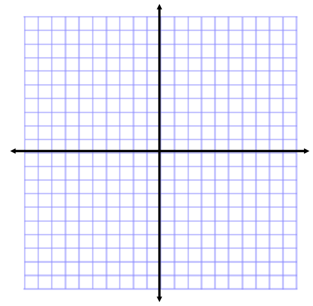
Given three vertices of a triangle: (x^1,y^1), (x^2,y^2), and (x^3,y^3), the coordinates of the centroid are the ***average*** of all of those points. Therefore, the coordinates of the centroid can be found by this rule:

$$\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$$

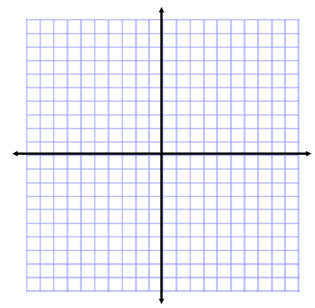
This helps to explain the fact that the centroid is the "center of gravity" of a triangle because it is *exactly* in the middle of a triangle.

Examples

1. Given ΔABC with coordinates A(0,0), B(4,0), and C(2,6), what are the coordinates of the centroid?



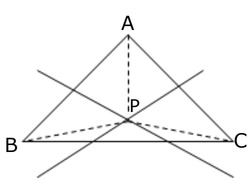
2. ΔABC has vertices A(-3,3), B(2,5), and C(4,-3). What are the coordinates of the centroid of ΔABC?



Concurrency of the Perpendicular Bisectors

The *perpendicular bisector* of a triangle is a line segment that is perpendicular (forms a right angle) and passes through the midpoint of a side of a triangle. There are three perpendicular bisectors in a triangle (one through each side).

The perpendicular bisectors of the three sides of a triangle are concurrent in a point that is equidistant (the same distance) from the vertices of the triangle. The point of concurrency of the perpendicular bisectors is known as the **circumcenter** of the triangle.

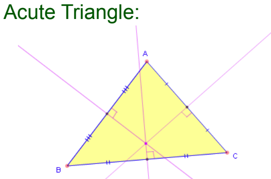


Since point P is the point of concurrency of the perpendicular bisectors, AP = BP = CP

Location of the Circumcenter

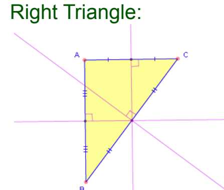
Unlike the centroid, the circumcenter is not always located inside the triangle. The location of the circumcenter depends on the type of triangle that we have.

Acute Triangle:



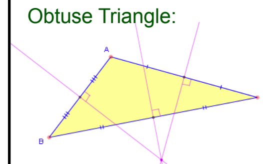
The circumcenter is located ***inside*** the triangle.

Right Triangle:



The circumcenter is located ***on*** the triangle.

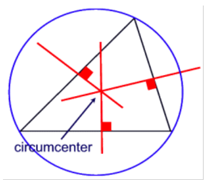
Obtuse Triangle:

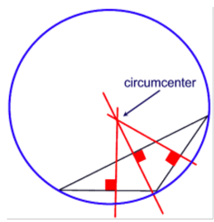


The circumcenter is located ***outside*** the triangle.

Properties of the Circumcenter

The circumcenter is the center of the circle that can be ***circumscribed*** around the triangle.





Examples

1. The perpendicular bisectors of ΔABC intersect at point P. If AP = 20 and BP = 2x+4, then what is the value of x?

2. The perpendicular bisectors of ΔABC intersect at point P. AP = 5 + x, BP = 10, and CP = 2y. Find x and y.

3. The perpendicular bisectors of ΔABC are concurrent at P. AP = 2x - 4, BP = y + 6, and CP = 12. Find x and y.

Concurrency of the Angle Bisectors

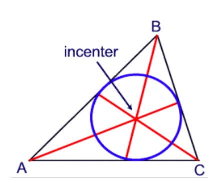
An *angle bisector* is a line segment with one endpoint on any vertex of a triangle that extends to the opposite side of the triangle and bisects the angle. There are three angle bisectors of a triangle.

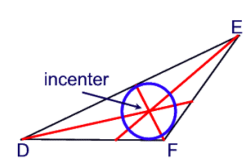
The three angle bisectors of a triangle are concurrent in a point equidistant from the sides of a triangle. The point of concurrency of the angle bisectors of a triangle is known as the **incenter** of a triangle.

The ***incenter*** will always be located **inside** the triangle.

Properties of the Incenter

The incenter is the center of a circle that is *inscribed* inside a triangle.





Concurrency of the Altitudes

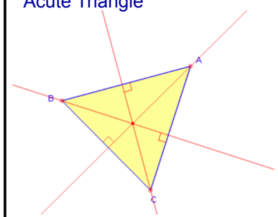
An *altitude* of a triangle is a line segment that is drawn from the vertex to the opposite side and is perpendicular to the side. There are three altitudes in a triangle.

The altitudes of a triangle, extended if necessary, are concurrent in a point called the **orthocenter** of the triangle.

Location of the Orthocenter

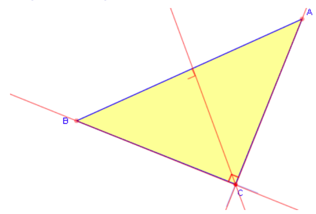
The orthocenter can fall in the interior of the triangle, on the side of the triangle, or in the exterior of the triangle.

Acute Triangle



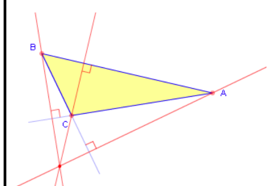
The orthocenter is located **inside** the triangle.

Right Triangle



The orthocenter is located **on** the right angle.

Obtuse Triangle



The orthocenter is located **outside** the triangle.